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Chaotic behavior of renormalization flow in a complex magnetic field

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It is demonstrated that decimation of the one-dimensional Ising model, with periodic boundary conditions, results in a nonlinear renormalization transformation for the couplings which can lead to chaotic behavior when the couplings are complex. The recursion relation for the couplings under decimation is equivalent to the logistic map, or more generally the Mandelbrot map. In particular, an imaginary external magnetic field can give chaotic trajectories in the space of couplings. The magnitude of the field must be greater than a minimum value which tends to zero as the critical point $T=0$ is approached, leading to a gap equation and an associated critical exponent which are identical to those of the Lee-Yang edge singularity in one dimension.

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The renormalization group has been developed into an immensely powerful tool for the analysis of physical theories near critical points and also for continuum field theories. There are by now various forms of the "renormalization group equation" which govern how physical amplitudes and couplings change under change of scale. Perhaps one of the most intuitively appealing is the version due to Wilson [1], motivated by a suggestion of Kadanov [2], involving "decimation."

In principle one can derive recursive formulas for the couplings of a theory, which dictate how they should change when the underlying lattice is decimated, so that the Hamiltonian involving the new couplings on the new lattice is the same as the Hamiltonian involving the old couplings on the old lattice, i.e., the partition function does not change under the simultaneous operations of decimation and redefinition of couplings. The recursive formulas for the couplings are in general nonlinear (indeed they are not invertible, so the transformation involved here is not a group but a semigroup).

Nonlinear recursive formulas are one of the central themes of study for chaos theory and one can pose the following question: can the renormalization transformation lead to chaotic behavior in the space of couplings? This possibility has been investigated before, and answered in the affirmative using numerical calculations in some specific models for which exact recursion relations can be obtained [3–6]. Here a simple model [the one-dimensional (1D) Ising model] will be analyzed analytically and it will be shown that this model also exhibits chaotic behavior in a surprisingly elegant manner. The analysis shows that the onset of chaotic behavior appears to be associated with the second order phase transition at $T=0$. It remains an open question as to whether this is a peculiarity of this model or a more general feature.

One severe problem in extracting general features is the paucity of models for which the recursion relations can be obtained exactly, and the one-dimensional Ising model—because of its simplicity—is one example for which progress can be made. Nevertheless, despite its simplicity, the results are startling enough to merit description.

It will be shown that the onset of chaotic behavior is brought about by extending the couplings of the theory to the complex plain. This is not a new idea in the analysis of such theories. Dyson [7] pointed out that one could learn something about the structure of quantum electrodynamics by considering imaginary electric charges, so that $\alpha=e^2/\hbar c<0$. Such a theory must be intrinsically unstable, and so amplitudes cannot be analytic at $\alpha=0$, hence perturbation theory must diverge and expansions in α are, at best, asymptotic. These ideas have been further developed by making α complex and there is by now a whole literature on complex analyticity and Borel summability (e.g., [8]). In statistical mechanics, extending the couplings to the complex plane is a key step in solving many two-dimensional models [9] and has led to some beautiful results concerning the analyticity of the partition function [10].

In this paper yet another example of the fascination of complex variables will be exhibited—by allowing the couplings of the one-dimensional Ising model to be complex, the recursive renormalization transformations can become chaotic. To exhibit this phenomenon, some well known features of the one-dimensional Ising model will be summarized and the recursion relations derived. It will then be shown that the recursion relation is nothing other than the logistic map, and chaos ensues.

Consider the one-dimensional Ising model on a periodic lattice of N sites [9]. The partition function is

$$Z_N=\sum_{\{\sigma\}}\exp\left[K\sum_{j=1}^{N}\sigma_j\sigma_{j+1}+h\sum_{j=1}^{N}\sigma_j\right],\qquad(1)$$

where $K=J/kT$ and $h=H/kT$, with J the spin coupling

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and H the external magnetic field (periodic boundary conditions require $\sigma_{N+1}\equiv\sigma_1$). $Z_N(K,h)$ can be conveniently expressed in terms of the transfer matrix

$$V=\begin{bmatrix} V_{++} & V_{+-} \\ V_{-+} & V_{--} \end{bmatrix}=\begin{bmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{bmatrix} \tag{2}$$

as $Z_N=\mathrm{Tr}\,V^N$.

Diagonalizing V gives the eigenvalues

$$\lambda_{\pm}=e^{K}\{\cosh h \pm(\sinh^2 h+e^{-4K})^{1/2}\} . \tag{3}$$

Thus

$$Z_N=\lambda_+^N\left[1+\left[\frac{\lambda_-}{\lambda_+}\right]^N\right] . \tag{4}$$

The recursive renormalization transformation is well known for the 1D Ising model [11]. It is obtained by asking can one find new couplings K' and h' such that

$$Z_{N/2}(K',h')=A^N Z_N(K,h) \tag{5}$$

gives the same physical amplitudes? (A is a normalization factor.)

Equation (5) is easily satisfied by demanding

$$\begin{bmatrix} e^{K'+h'} & e^{-K'} \\ e^{-K'} & e^{K'-h'} \end{bmatrix}=A^2\begin{bmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{bmatrix}^2 , \tag{6}$$

giving the recursive formulas

$$e^{2h'}=e^{2h}\frac{\cosh(2K+h)}{\cosh(2K-h)} , \tag{7}$$

$$e^{4K'}=\frac{\cosh(4K)+\cosh(2h)}{2\cosh^2(h)} . \tag{8}$$

The normalization factor A is unimportant for the present analysis.

The combination $e^{4K'}\sinh^2(h')=e^{4K}\sinh^2(h)$ is a renormalization transformation invariant. This is not unexpected since the magnetization per site, in the thermodynamic limit, is

$$M_{N\to\infty}=-\frac{\partial \ln Z_N}{N\partial h}=-\frac{e^{2K}\sinh(h)}{[1+e^{4K}\sinh^2(h)]^{1/2}} , \tag{9}$$

which is a physical quantity.

All these facts about the one-dimensional Ising model are well known [9] and are included only for completeness. It will now be shown that the recursion relations (7) and (8) are equivalent to the logistic map and, for certain (complex) values of couplings, give rise to chaotic behavior.

Define

$$m=1+e^{4K}\sinh^2(h) , \tag{10}$$

which is a renormalization transformation invariant, $m=m'$. It is now only necessary to consider one of Eqs. (8) and (7) as the existence of the invariant, m, makes one of them redundant.

Eliminating h from (8) using (10) gives

$$e^{4K'}-1=\frac{1}{4}\frac{(e^{4K}-1)^2}{[(e^{4K}-1)+m]} . \tag{11}$$

Now replace K with a new variable

$$x=-\frac{m}{e^{4K}-1} \tag{12}$$

with $-\infty<x<0$ for $m>0$ and $K>0$. The recursion relation (11) now becomes

$$x'=4x(1-x) , \tag{13}$$

which is the logistic map.

For $0<x<1$, the recursion relation (13) leads to chaotic behavior, as is easily seen by defining $x=\sin^2(\pi\psi)$, $0<\psi<\frac{1}{2}$ (see, e.g., [12]) giving

$$\sin(\pi\psi')=\sin(2\pi\psi) . \tag{14}$$

Writing ψ in binary form, we see that the iterative map merely shifts all bits one step to the left and throws away the integral part, leaving the fractional part behind. For an initial value of ψ which is rational this will lead to a periodic orbit, but for a starting value of ψ which is irrational, the process never repeats and ψ jumps around chaotically. Since the irrational numbers have a greater cardinality than the rational numbers, almost all initial values lead to chaotic motion.

For real values of the couplings, $m>1$ and $-\infty<x<0$. Chaotic trajectories require

$$m=1+e^{4K}\sinh^2(h)<0 . \tag{15}$$

For example, if $K>0$ is real and h is pure imaginary ($h=i\theta$), then $m<0$ for $\sin^2\theta>e^{-4K}$ and $x=(e^{4K}\sin^2\theta-1)/(e^{4K}-1)$ lies between 0 and 1 for $e^{4K}>1/\sin^2\theta$.

The region of chaotic flow is shown in Fig. 1. Note that not all points above the line $x=0$ lead to chaotic trajectories, only those with irrational ψ. Indeed there is an infinite number of periodic trajectories as well as chaotic ones. For example, there are lines of unstable fixed points (period 1) at $x=0$ and $x=\frac{3}{4}$ and the values $x=(5\pm\sqrt{10})/8$ give orbits of period 2, etc.

For finite K there is a gap and a small imaginary magnetic field is not sufficient to induce chaos, but as K in-

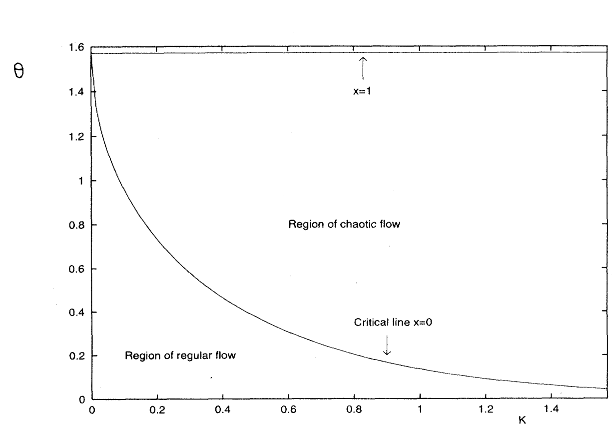

FIG. 1. Critical line in the K-θ plane. The renormalization flow is regular below the critical line $x=0$ and chaotic above it for all values of ψ which are irrational [$x=\sin^2(\pi\psi)$].

creases ($T \rightarrow 0$) this gap reduces to zero. Following Baxter [9], define $t=e^{-2K}$ with $t \rightarrow 0$ being the critical point, then near $t=0$ the line separating chaotic from regular flow is given by

$$t^2 \sim \theta^2 \Longrightarrow \theta \sim t \ . \tag{16}$$

If we define a critical exponent Δ such that the critical value of θ is

$$\theta \sim t^{\Delta} , \tag{17}$$

then one obtains $\Delta=1$ for the one-dimensional Ising model, which is just the critical exponent for the Lee-Yang edge singularity in this model. It is not difficult to see that the critical line $e^{4K}\sin^2\theta=1$ is related to the Lee-Yang zeros of the partition function in the complex h plane. The equation $Z_N(K,h)=0$ has N roots, for real K,

$$Z_N=(\lambda_+)^N+(\lambda_-)^N=0 \Longleftrightarrow \lambda_+=e^{iq\pi/N}\lambda_- , \tag{18}$$

where $-N<q \le N$ is odd. Using the explicit form of the eigenvalues, (3), this leads to

$$\cos\left[\frac{q\pi}{2N}\right][1+e^{4K}\sinh^2(h)]^{1/2} + i \sin\left[\frac{q\pi}{2N}\right]\cosh(h)=0 \ . \tag{19}$$

The square of this equation can be rearranged to give

$$\tan^2(\theta_q)=\frac{t^2}{(1-t^2)}\frac{1}{\cos^2\left[\frac{q\pi}{2N}\right]} , \tag{20}$$

where $t=e^{-2K}$ as before and θ_q are the N roots in the rotated complex h plane, $h=i\theta$. Since $0<t<1$, the N values of θ_q are all real. In the thermodynamic limit $N\rightarrow\infty$, the zeros are all degenerate and $\sin(\theta_q)=e^{-2K}$, which is exactly the equation for the critical line. Thus the critical line coincides with the Lee-Yang zeros in the thermodynamic limit. For finite N, $\theta_q \rightarrow 0$ for all q as $K\rightarrow\infty$, and $\theta_q=\pi/2$ for all q when $K=0$.

One can obtain further insights by allowing K to become complex. Define $x=-z/4+\frac{1}{2}$ in Eq. (13) to give

$$z'=z^2-2 \ . \tag{21}$$

This is the Mandelbrot map $z'=z^2+c$ for complex z, with $c=-2$. One can have divergence or convergence depending on c and the initial choice of z. The values of c for which the iterates of the starting point $z=0$ stay bounded is the Mandelbrot set and clearly $c=-2$ is an element of this set. The Julia set for a given value of c is the set of points in the complex z plane which stay within a bounded region upon repeated iteration of the Mandelbrot map (strictly speaking this is the filled in Julia set J_c—the Julia set J is actually the boundary of this set). The set J is generated by the inverse set of the unstable fixed points. For $c=-2$ these are $z=-1$ and $z=2$ and so the earlier analysis of the logistic map tells us that the inverse iterates generate a dense set of points in the segment of the real axis lying between -2 and $+2$ (this corresponds to $0<x<1$ in the previous notation). Thus the filled in Julia set is just the segment of the real axis with $-2<z<2$. This analysis shows that the forward iterations send $|z|$ to infinity if the temperature has an imaginary component or if the magnetic field has both real and imaginary parts nonzero (this latter possibility would result in m having nonzero imaginary part and thus so would z). Thus chaotic trajectories occur only for real K and pure imaginary h. The behavior of the Julia set under iteration is shown in Fig. 2 for $c=-2$.

An obvious question is how generic is this behavior? For a general Hamiltonian when is it possible to obtain chaotic behavior in some region of (complex) coupling space? For the moment this question must remain unanswered, but a few comments should be made. Feigenbaum was aware of the universality in chaos [13]. Near an extremum any nonlinear map (with nonvanishing second derivative) can be put into the form (13) with the number 4 replaced, in general, by a parameter, λ. Thus

$$x'=\lambda x(1-x) \tag{22}$$

is generic, but whether or not one has chaotic behavior depends on the value of λ and the initial value of x. I do not know of any reason why λ has the rather special value of 4 for the one-dimensional Ising model. More generally, Feigenbaum has shown [13] that the properties of a general nonlinear map

$$x'=\lambda f(x) \tag{23}$$

are independent of the exact form of $f(x)$ near a maximum. For any particular model, the value of λ would have to be calculated *ab initio* and I know of no way of deciding in advance whether or not chaotic flow would result. For the 1D Ising model it is clear that the onset of chaotic trajectories is related to the second order phase

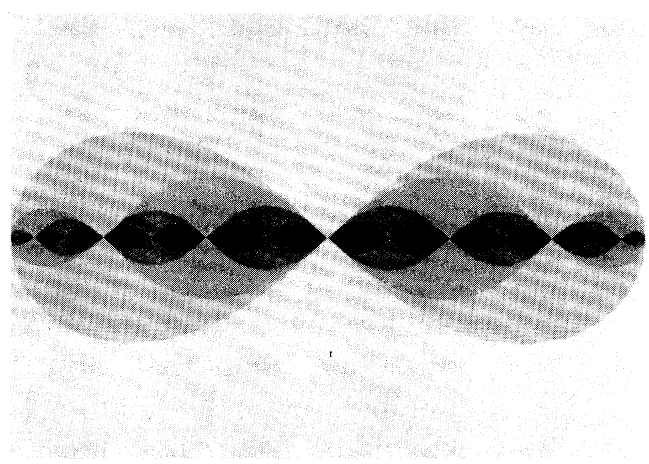

FIG. 2. A representation of the Julia set in the complex z plane for $c=-2$. The Julia set itself is the real line segment $-2<\mathrm{Re}(z)<2$ which is the width of the diagram. The contours depict the rate at which a point is repelled from the Julia set; the darker the contour the less rapid the expulsion. The picture was generated using the program FRACTINT produced by the Stone Soup Group.

transition at $T=0$ and the existence of the Lee-Yang edge singularity. However, other models show chaotic recursive maps for real values of the couplings which are not related to second order phase transitions—rather they are related to frustration and glasslike structures [3]. Thus it does not appear that a second order phase transition is a prerequisite for chaos, but neither are spin glasses a prerequisite. Unfortunately the number of models for which the recursion relations are known exactly is rather few. In particular the recursion relations for the 2D Ising model are not known, so it is not possible at this stage to say whether or not the two-dimensional Lee-Yang edge singularity is related to chaotic trajectories.

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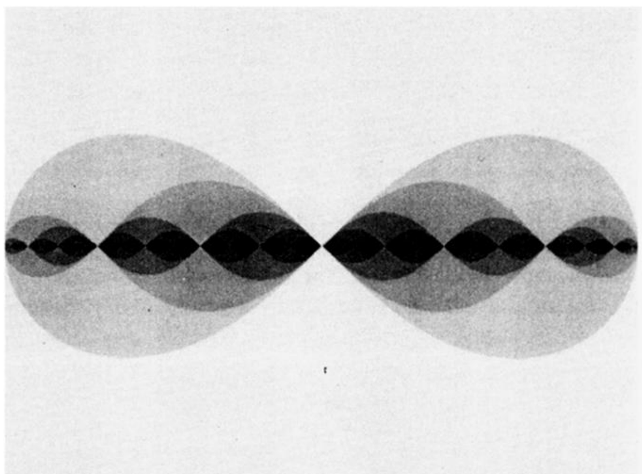

FIG. 2. A representation of the Julia set in the complex z plane for $c = -2$. The Julia set itself is the real line segment $-2 < \mathrm{Re}(z) < 2$ which is the width of the diagram. The contours depict the rate at which a point is repelled from the Julia set; the darker the contour the less rapid the expulsion. The picture was generated using the program FRACTINT produced by the Stone Soup Group.